

AN INTERMEDIATE-GROWTH GROUP WITH INFINITE NUCLEAR DIMENSION

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ABSTRACT. Let G_{012} be the first Grigorchuk group and let X be its orbit of the ray $111\cdots$ in the binary-tree boundary. We prove that the restricted permutational wreath product $W = \mathbb{Z} \wr_X G_{012}$ has infinite nuclear dimension for both its full and reduced group C^* -algebras. The group is finitely generated and has intermediate word growth by a theorem of Bartholdi and Erschler. The dimension argument uses finite-action quotients and commutative corners in their group C^* -algebras. More generally, it applies to any permutational wreath product whose lamp group maps nontrivially to $\mathbb{Z}$ and whose acting set has finite equivariant images of unbounded cardinality. No growth assumption is needed for this general obstruction.

1. THE RESULT

All groups in this paper are discrete. We write $C^*(H)$ and $C_r^*(H)$ for the full and reduced group C^* -algebras. Nuclear dimension takes values in $\{0, 1, 2, \dots, \infty\}$. In particular, a nonnuclear algebra has infinite nuclear dimension.

Let $T = \{0, 1\}^*$ be the rooted binary tree. We use right actions, so that $v(gh) = (vg)h$. The first Grigorchuk group G_{012} is the subgroup of the automorphism group of T generated by the recursively defined automorphisms

$$a = (1, 1)\sigma, \quad b = (a, c), \quad c = (a, d), \quad d = (1, b),$$

where σ exchanges the two first-level vertices. For example, $(0v)b = 0(va)$ and $(1v)b = 1(vc)$. These recursions define actual automorphisms on finite words and hence on the boundary; they are not being used as an abstract recursive presentation.

Put $\rho = 111\cdots$ and $X = \rho G_{012}$. For a right H -set X , the restricted permutational wreath product with integer lamps is

$$\mathbb{Z} \wr_X H = \mathbb{Z}^{(X)} \rtimes H, \quad (g \cdot f)(x) = f(xg),$$

where $\mathbb{Z}^{(X)}$ denotes finitely supported functions and $(f, g)(h, k) = (f + g \cdot h, gk)$. The formula is a left action on functions: $g \cdot (h \cdot f) = (gh) \cdot f$.

Theorem 1.1. *The group*

$$W = \mathbb{Z} \wr_X G_{012}, \quad X = (111\cdots)G_{012},$$

is finitely generated and has word growth equivalent to $\exp(n^\alpha \log n)$, where $\alpha = \log 2 / \log(2/\eta)$ and $\eta \in (0, 1)$ is the root of $\eta^3 + \eta^2 + \eta - 2 = 0$. In particular, its growth is subexponential and nonpolynomial. Its full and reduced group C^ -algebras are canonically isomorphic and nuclear, and*

$$\dim_{\text{nuc}} C^*(W) = \dim_{\text{nuc}} C_r^*(W) = \infty.$$

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The group in Theorem 1.1 is the established cyclic-lamp example of Bartholdi and Erschler [1, Theorem 5.3]. Their theorem supplies the growth assertion. The purpose of this note is to record its infinite nuclear dimension by combining quotients obtained from finite permutation actions with the permanence properties of Winter and Zacharias [7].

Infinite nuclear dimension for finitely generated amenable groups was already known. In particular, Eckhardt and McKenney [3, Theorem 5.1] record the result for $\mathbb{Z} \wr \mathbb{Z}$, attributing it to the work of Giol and Kerr [5]. Eckhardt and Wu [4, Example 6.1] give a proof using virtually abelian quotients for that group, and their Theorem 6.4 treats a broader class of ordinary wreath products. The obstruction using finite-action quotients here is closely related to that argument; we apply it to the boundary-orbit permutational wreath product of [1]. The ordinary-wreath-product theorem does not directly apply to this action.

More recently, Chan, Lippert, Moutzouris, and Weld [2, Theorem 4.11] prove that the nuclear dimension of the group C^* -algebra of a finitely generated virtually abelian group equals its Hirsch length. Their result gives an alternative route to the quotient lower bounds below. We retain an elementary averaging-corner proof of the needed lower bound. No claim is made to a new group construction, to the general strategy using virtually abelian quotients, or to an exhaustive determination of priority for the particular intermediate-growth consequence.

2. QUOTIENTS FROM FINITE PERMUTATION ACTIONS

We use three facts from [7, Propositions 2.3(iv), 2.4, and 2.5]: nuclear dimension does not increase under quotients or passage to hereditary subalgebras, and

$$\dim_{\text{nuc}} C_0(Z) = \dim Z$$

for a second-countable locally compact Hausdorff space Z . Corners by projections are hereditary subalgebras; no fullness assumption is needed for the inequality we use.

Lemma 2.1. *Let Y be a nonempty finite set, let $F \leq \text{Sym}(Y)$, and let F act on $\mathbb{Z}^Y$ by permutation of coordinates. Then*

$$\dim_{\text{nuc}} C^*(\mathbb{Z}^Y \rtimes F) \geq |Y|.$$

Proof. Set $A = C(\mathbb{T}^Y)$ and $B = A \rtimes F$, with the full crossed product. The universal relations identify B with $C^*(\mathbb{Z}^Y \rtimes F)$: the commuting lamp unitaries represent $\mathbb{Z}^Y$, and the implementing unitaries impose precisely the semidirect-product conjugation relations.

Write u_h for these implementing unitaries and put

$$e = \frac{1}{|F|} \sum_{h \in F} u_h, \quad E_F(a) = \frac{1}{|F|} \sum_{h \in F} \alpha_h(a).$$

The finite-group identities give $e = e^* = e^2$ and $u_h e = e = e u_h$. Expanding the sums and using covariance gives $e a e = E_F(a) e$. Thus the map

$$A^F \longrightarrow e B e, \quad a \longmapsto a e,$$

is a surjective $*$ -homomorphism. Indeed, invariant functions commute with e , and compressing the dense algebra of sums $\sum_h a_h u_h$ gives precisely elements of this form.

This map is injective. Represent A faithfully and form the regular covariant representation on its representation space tensored with $\ell^2(F)$. Compression to the identity coordinate sends $a e$ to the faithful representation of $a/|F|$. Therefore $a e = 0$ in the full crossed product implies $a = 0$. It follows that

$$(1) \quad e B e \cong A^F \cong C(\mathbb{T}^Y / F).$$

The finite orbit space $Z = \mathbb{T}^Y/F$ is compact Hausdorff and metrizable. For instance, average a metric on the torus over F and use the minimum distance between the two finite orbits as the quotient metric.

Let $m = |Y|$. Choose m pairwise disjoint nondegenerate closed proper arcs $K_y \subset \mathbb{T}$, and set $K = \prod_{y \in Y} K_y$. Then $K \cong [0, 1]^m$. Every nonidentity coordinate permutation moves at least one arc to a disjoint arc, so $K \cap hK = \emptyset$ for $h \neq 1$. Consequently the quotient map is injective on K and identifies it homeomorphically with a closed subset of Z . Restriction, together with Tietze extension, gives a surjection

$$C(Z) \longrightarrow C(K) \cong C([0, 1]^m).$$

The three dimension facts stated above now yield

$$m = \dim_{\text{nuc}} C([0, 1]^m) \leq \dim_{\text{nuc}} (eBe) \leq \dim_{\text{nuc}} B.$$

□

The following standard amenability criterion is due to Lance [6]. We include a proof, together with the equality of the two completions in the amenable case, that does not require countability.

Lemma 2.2. *Let H be an arbitrary discrete group. Then $C_r^*(H)$ is nuclear if and only if H is amenable. If H is amenable, the canonical map $C^*(H) \rightarrow C_r^*(H)$ is an isomorphism. These assertions require no countability assumption.*

Proof. We use the tensor-norm characterization of nuclearity: the minimal and maximal tensor norms agree with every C^* -algebra. Write $A = C_r^*(H) \subset B(\ell^2(H))$, let λ be the left regular representation, and define $r(g)\delta_h = \delta_{hg^{-1}}$. The linear unitary $\delta_h \mapsto \delta_{h^{-1}}$ conjugates $\lambda(g)$ to $r(g)$, so $R = C^*(r(H))$ is a copy of A . These representations commute and give

$$\Pi : A \otimes_{\max} R \longrightarrow B(\ell^2(H)), \quad \Pi(a \otimes b) = ab.$$

If A is nuclear, the vector functional

$$\omega(z) = \langle \Pi(z)\delta_1, \delta_1 \rangle$$

is therefore a state on $A \otimes_{\min} R$. By injectivity of the minimal tensor product, this algebra embeds unitally in $B(\ell^2(H)) \otimes_{\min} R$. Extend ω to a state Ω on the latter algebra.

For every $g \in H$, the unitary $u_g = \lambda(g) \otimes r(g)$ has $\Omega(u_g) = 1$, since $\lambda(g)r(g)\delta_1 = \delta_1$. Thus both $\Omega((u_g - 1)^*(u_g - 1))$ and $\Omega((u_g - 1)(u_g - 1)^*)$ vanish. State Cauchy–Schwarz shows that multiplication on either side by u_g or u_g^* leaves Ω unchanged. In particular, $\Omega(u_g z u_g^*) = \Omega(z)$.

For a bounded function f on H , let M_f be its diagonal multiplication operator. The formula

$$m(f) = \Omega(M_f \otimes 1)$$

defines a positive normalized functional. Conjugation by u_g sends $M_f \otimes 1$ to $M_{L_g f} \otimes 1$, where $(L_g f)(h) = f(g^{-1}h)$. Hence m is left invariant and H is amenable. No cardinality restriction has entered.

Conversely, let m be a left-invariant mean. We first obtain finitely supported probability measures μ with $\|g\mu - \mu\|_1$ simultaneously small on an arbitrary finite set $E \subset H$, where $(g\mu)(h) = \mu(g^{-1}h)$. Indeed, zero lies in the norm closure of the convex set

$$\{(g\mu - \mu)_{g \in E} : \mu \text{ is a finitely supported probability measure}\} \subset \bigoplus_{g \in E} \ell^1(H).$$

Otherwise real Hahn–Banach separation would give bounded real functions f_g and $c > 0$ such that

$$\sum_{g \in E} (f_g(gh) - f_g(h)) \geq c \quad \text{for every } h \in H,$$

by testing point masses. Applying m contradicts left invariance. Directing these choices by finite sets and error tolerances yields a net (μ_i) with $\|g\mu_i - \mu_i\|_1 \rightarrow 0$ for each g . The unit vectors $\xi_i = \sqrt{\mu_i}$ satisfy

$$\|\lambda(g)\xi_i - \xi_i\|_2^2 \leq \|g\mu_i - \mu_i\|_1 \rightarrow 0.$$

For any unitary representation π , the isometries $v \mapsto v \otimes \xi_i$ asymptotically intertwine $\pi(g)$ with $\pi(g) \otimes \lambda(g)$. The unitary

$$U(v \otimes \delta_h) = \pi(h)^{-1}v \otimes \delta_h$$

conjugates $\pi(g) \otimes \lambda(g)$ to $1 \otimes \lambda(g)$. Consequently $\|\pi(z)\| \leq \|\lambda(z)\|$ for every finite group-ring sum z . Taking the supremum over π proves equality of the full and reduced norms and the canonical isomorphism. Only nets over finite sets and tolerances were used.

To prove nuclearity, put $F_i = \text{supp } \xi_i$ and let $\Phi_i : A \rightarrow M_{F_i}$ be compression. Define

$$\Psi_i(e_{x,y}) = \xi_i(x)\xi_i(y)\lambda(xy^{-1}).$$

The row $(\xi_i(x)\lambda(x))_{x \in F_i}$ implements Ψ_i and has product with its adjoint equal to one. Thus Φ_i and Ψ_i are unital completely positive. Their composite satisfies

$$\Psi_i\Phi_i(\lambda(g)) = \left(\sum_y \xi_i(gy)\xi_i(y) \right) \lambda(g) = \left(1 - \frac{1}{2}\|\lambda(g)\xi_i - \xi_i\|_2^2 \right) \lambda(g).$$

It converges in norm to $\lambda(g)$, and contractivity and density give point-norm convergence to the identity of A .

For completeness, let B be any C^* -algebra and $z = \sum_{k=1}^q a_k \otimes b_k \in A \odot B$. Completely positive tensor functoriality and equality of the two tensor norms for matrix algebras give

$$\begin{aligned} \|(\Psi_i\Phi_i \otimes \text{id})(z)\|_{\max} &\leq \|(\Phi_i \otimes \text{id})(z)\|_{M_{F_i} \otimes_{\max} B} \\ &= \|(\Phi_i \otimes \text{id})(z)\|_{M_{F_i} \otimes_{\min} B} \leq \|z\|_{\min}. \end{aligned}$$

Meanwhile the maximal norm of the difference from z is bounded by $\sum_k \|\Psi_i\Phi_i(a_k) - a_k\| \|b_k\|$, which tends to zero. Hence $\|z\|_{\max} \leq \|z\|_{\min}$; the reverse inequality always holds. This proves nuclearity of A , and therefore of its canonically isomorphic full completion. $\square$

Proposition 2.3. *Suppose a discrete group H admits epimorphisms*

$$H \longrightarrow \mathbb{Z}^{Y_j} \rtimes F_j, \quad F_j \leq \text{Sym}(Y_j),$$

where the Y_j are nonempty finite sets with unbounded cardinalities. Then

$$\dim_{\text{nuc}} C^*(H) = \dim_{\text{nuc}} C_r^*(H) = \infty.$$

Proof. A group epimorphism induces a surjection of full group C^* -algebras, by their universal property. Lemma 2.1 and quotient permanence give

$$\dim_{\text{nuc}} C^*(H) \geq |Y_j| \quad \text{for every } j,$$

so the full algebra has infinite nuclear dimension.

If the reduced algebra had finite nuclear dimension, it would be nuclear. Lemma 2.2 would then make H amenable and identify its full and reduced completions, contradicting the full-algebra conclusion. Thus the reduced dimension is also infinite. Only full-algebra quotient maps have been used. $\square$

3. THE GROUP AND ALL ITS FINITE-ACTION QUOTIENTS

Let $G = G_{012}$ and retain the orbit $X = \rho G$. The four tree generators are involutions, by induction on the length of tree words in their defining recursions. If $t = (\delta_\rho, 1) \in W$, then

$$S = \{(0, a), (0, b), (0, c), (0, d), t, t^{-1}\}$$

is a finite symmetric generating set. Indeed, conjugating t by $(0, g)$ gives the lamp $\delta_{\rho g^{-1}}$, and these are all the basis lamps because X is an orbit.

We verify level transitivity directly, to ensure that no lamp coordinates are lost in the quotients. Let $H_1 \leq G$ fix both first-level vertices. Its two section maps into G are homomorphisms. The six elements b, c, d, aba, aca, ada belong to H_1 and have section pairs

$$(a, c), (a, d), (1, b), (c, a), (d, a), (b, 1),$$

respectively. Each section map is therefore onto G .

The generator a acts transitively on the first level. Suppose G is transitive on suffixes of length n . To send iu to ju , first apply 1 or a to match the first letter. Choose an element of G sending u to v , and lift it through the j -section map of H_1 . The lift fixes the first letter and performs the required change of suffix. Induction proves transitivity on every level

$$Y_n = \{0, 1\}^n \quad (n \geq 1).$$

Let F_n be the image of G in its right permutation action on Y_n , and let $r_n : G \rightarrow F_n$ be the resulting epimorphism. The prefix map $p_n : X \rightarrow Y_n$ satisfies

$$p_n(xg) = p_n(x)r_n(g).$$

It is onto, since the prefix 1^n of ρ has the whole level as its orbit. In particular, X is infinite.

For $f \in \mathbb{Z}^{(X)}$, define

$$(\Sigma_n f)(y) = \sum_{x \in p_n^{-1}(y)} f(x).$$

These sums are finite because f has finite support. Equip $\mathbb{Z}^{Y_n}$ with the action $(h \cdot u)(y) = u(yh)$. Reindexing by $z = xg$ gives

$$\Sigma_n(g \cdot f)(y) = \sum_{p_n(z)=yr_n(g)} f(z) = (r_n(g) \cdot \Sigma_n f)(y).$$

It follows that

$$(2) \quad q_n : W \longrightarrow \mathbb{Z}^{Y_n} \rtimes F_n, \quad q_n(f, g) = (\Sigma_n f, r_n(g))$$

is a homomorphism. It is onto: choose one preimage $x_y \in X$ of each $y \in Y_n$, lift a lamp u by $\sum_y u(y)\delta_{x_y}$, and lift the desired permutation independently through r_n .

Thus (2) gives an epimorphism for every $n \geq 1$, with $|Y_n| = 2^n$. Proposition 2.3 already proves both asserted infinite dimension values.

4. GROWTH AND THE NUCLEAR COMPLETION

Let $\eta \in (0, 1)$ be the real root of $\eta^3 + \eta^2 + \eta - 2 = 0$, and put

$$\alpha = \frac{\log 2}{\log(2/\eta)} \in (0, 1).$$

Bartholdi and Erschler's Theorem 5.3 applies to $L_0 = \mathbb{Z}$ and $L_1 = \mathbb{Z} \wr_X G_{012}$, with this same boundary orbit X [1, §§2, 3.1, 5]. For the word growth $v(n) = |B_S(n)|$, it supplies the two-sided comparison

$$v(n) \sim \exp(n^\alpha \log n).$$

Here $f \preceq h$ means $f(n) \leq h(Cn)$ eventually for some $C > 0$, with integer rounding understood; $\sim$ means both comparisons. These comparisons are unchanged by replacing one finite generating set with another. Monotonicity of v , with a fixed rescaling and integer rounding, gives constants $c, C > 0$ such that eventually

$$\exp(cn^\alpha \log n) \leq v(n) \leq \exp(Cn^\alpha \log n).$$

For the lower estimate, if $\exp(k^\alpha \log k) \leq v(Lk)$ eventually, take $k = \lfloor n/L \rfloor$ and use $Lk \leq n$; enlarging L and the threshold absorbs rounding. The upper estimate is the direct rescaled comparison. Since $0 < \alpha < 1$, the upper estimate gives $\log v(n)/n \rightarrow 0$. For every $d \geq 0$ and $M > 0$, the lower estimate gives

$$\log \frac{v(n)}{Mn^d} \geq (cn^\alpha - d) \log n - \log M \rightarrow \infty.$$

Thus the growth is subexponential and exceeds every proposed polynomial bound eventually.

We include the completion and nuclearity arguments to distinguish this example from a non-nuclear source of infinite nuclear dimension. Subexponential growth implies

$$\liminf_{m \rightarrow \infty} \frac{v(m+1)}{v(m)} = 1,$$

since an eventual lower bound strictly above one would force exponential growth. Choose increasing m_j along which the ratio tends to one, and put $D_j = B_S(m_j)$. For $s \in S$,

$$\frac{|sD_j \triangle D_j|}{|D_j|} \leq 2 \frac{v(m_j+1) - v(m_j)}{v(m_j)} \rightarrow 0.$$

A word decomposition gives the same assertion for every fixed group element. Thus (D_j) is a left Følner sequence, and a weak-* cluster point of its averaging functionals is an invariant mean.

Lemma 2.2 now identifies the two completions canonically and proves their nuclearity. Together with the dimension lower bounds in the preceding sections, this completes the proof of Theorem 1.1.

5. THE TECHNICAL SCOPE OF THE OBSTRUCTION

Theorem 5.1. *Let A and G be discrete groups, let X be a right G -set, and suppose that:*

- (1) *there is a nonzero homomorphism $A \rightarrow \mathbb{Z}$;*
- (2) *X has G -equivariant surjections onto nonempty finite G -sets Y_j with unbounded cardinalities.*

Then, for the restricted permutational wreath product $V = A \wr_X G$,

$$\dim_{\text{nuc}} C^*(V) = \dim_{\text{nuc}} C_r^*(V) = \infty.$$

Proof. A nonzero subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$, so the first hypothesis supplies an epimorphism $\chi : A \rightarrow \mathbb{Z}$. For each finite equivariant quotient $p : X \rightarrow Y$, let F be the image of G in $\text{Sym}(Y)$. If $f : X \rightarrow A$ is finitely supported, put

$$(\Sigma_\chi f)(y) = \sum_{x \in p^{-1}(y)} \chi(f(x)).$$

This is a homomorphism from $A^{(X)}$ to $\mathbb{Z}^Y$, even when A is nonabelian. Equivariance follows by exactly the reindexing used in (2). It gives a homomorphism $V \rightarrow \mathbb{Z}^Y \rtimes F$. This map is onto: use surjectivity of χ and one preimage in each fiber to lift the lamp coordinates, and lift a permutation through $G \rightarrow F$. Proposition 2.3 applies. $\square$

The general theorem uses neither finite generation, transitivity of X , faithfulness of the original action, amenability, nor a growth bound. These extra properties place the concrete example in the intermediate-growth class; they do not drive the dimension obstruction. For a finitely generated lamp group, the first hypothesis is equivalent to its abelianization having positive torsion-free rank. The method needs unbounded finite action images and integer lamp quotients to produce commutative corners of unbounded dimension. These are sufficient conditions for this argument, not necessary conditions for infinite nuclear dimension. No conclusion about actions lacking such images, or lamps with finite abelianization, is asserted.

USE OF AI TOOLS

OpenAI tools were used in developing and checking the argument, locating references, and preparing the manuscript.

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